Separación

1. From $dy = \sin 5x\, dx$ we obtain $y = -\frac{1}{5}\cos 5x + c$.
2. From $dy = (x+1)^2\, dx$ we obtain $y = \frac{1}{3}(x+1)^3 + c$.
3. From $dy = -e^{-3x}\, dx$ we obtain $y = \frac{1}{3}e^{-3x} + c$.
4. From $\dfrac{1}{(y-1)^2}\, dy = dx$ we obtain $-\dfrac{1}{y-1} = x + c$ or $y = 1 - \dfrac{1}{x+c}$.
5. From $\dfrac{1}{y}\, dy = \dfrac{4}{x}\, dx$ we obtain $\ln|y| = 4\ln|x| + c$ or $y = c_1 x^4$.
6. From $\dfrac{1}{y^2}\, dy = -2x\, dx$ we obtain $-\dfrac{1}{y} = -x^2 + c$ or $y = \dfrac{1}{x^2 + c_1}$.
7. From $e^{-2y} dy = e^{3x} dx$ we obtain $3e^{-2y} + 2e^{3x} = c$.
8. From $ye^y dy = \left(e^{-x} + e^{-3x}\right) dx$ we obtain $ye^y - e^y + e^{-x} + \dfrac{1}{3}e^{-3x} = c$.
9. From $\left(y + 2 + \dfrac{1}{y}\right) dy = x^2 \ln x\, dx$ we obtain $\dfrac{y^2}{2} + 2y + \ln|y| = \dfrac{x^3}{3}\ln|x| - \dfrac{1}{9}x^3 + c$.
10. From $\dfrac{1}{(2y+3)^2}\, dy = \dfrac{1}{(4x+5)^2}\, dx$ we obtain $\dfrac{2}{2y+3} = \dfrac{1}{4x+5} + c$.
11. From $\dfrac{1}{\csc y}\, dy = -\dfrac{1}{\sec^2 x}\, dx$ or $\sin y\, dy = -\cos^2 x\, dx = -\frac{1}{2}(1 + \cos 2x)\, dx$ we obtain $-\cos y = -\frac{1}{2}x - \frac{1}{4}\sin 2x + c$ or $4\cos y = 2x + \sin 2x + c_1$.
12. From $2y\, dy = -\dfrac{\sin 3x}{\cos^3 3x}\, dx$ or $2y\, dy = -\tan 3x \sec^2 3x\, dx$ we obtain $y^2 = -\frac{1}{6}\sec^2 3x + c$.
13. From $\dfrac{e^y}{\left(e^y+1\right)^2}\, dy = \dfrac{-e^x}{\left(e^x+1\right)^3}\, dx$ we obtain $-\left(e^y+1\right)^{-1} = \frac{1}{2}\left(e^x+1\right)^{-2} + c$.
14. From $\dfrac{y}{(1+y^2)^{1/2}}\, dy = \dfrac{x}{(1+x^2)^{1/2}}\, dx$ we obtain $\left(1+y^2\right)^{1/2} = \left(1+x^2\right)^{1/2} + c$.

Comprobacion

writing it in the form $(v + uv - ue^{u})(du/dv) + u = 0$, we see that it is nonlinear in u.

11. From $y = e^{-x/2}$ we obtain $y' = -\frac{1}{2}e^{-x/2}$. Then $2y' + y = -e^{-x/2} + e^{-x/2} = 0$.

12. From $y = \frac{6}{5} - \frac{6}{5}e^{-20t}$ we obtain $dy/dt = 24e^{-20t}$, so that

$$\frac{dy}{dt} + 20y = 24e^{-20t} + 20\left(\frac{6}{5} - \frac{6}{5}e^{-20t}\right) = 24.$$

13. From $y = e^{3x}\cos 2x$ we obtain $y' = 3e^{3x}\cos 2x - 2e^{3x}\sin 2x$ and $y'' = 5e^{3x}\cos 2x - 12e^{3x}\sin 2x$, so that $y'' - 6y' + 13y = 0$.

14. From $y = -\cos x\ln(\sec x + \tan x)$ we obtain $y' = -1 + \sin x\ln(\sec x + \tan x)$ and $y'' = \tan x + \cos x\ln(\sec x + \tan x)$. Then $y'' + y = \tan x$.

15. The domain of the function, found by solving $x + 2 \geq 0$, is $[-2, \infty)$. From $y' = 1 + 2(x+2)^{-1/2}$ we have

$$\begin{aligned}(y - x)y' &= (y - x)[1 + (2(x+2)^{-1/2}]\\ &= y - x + 2(y - x)(x+2)^{-1/2}\\ &= y - x + 2[x + 4(x+2)^{1/2} - x](x+2)^{-1/2}\\ &= y - x + 8(x+2)^{1/2}(x+2)^{-1/2} = y - x + 8.\end{aligned}$$

Factor integrante

1. For $y' - 5y = 0$ an integrating factor is $e^{-\int 5\,dx} = e^{-5x}$ so that $\frac{d}{dx}\left[e^{-5x}y\right] = 0$ and $y = ce^{5x}$ for $-\infty < x < \infty$.

2. For $y' + 2y = 0$ an integrating factor is $e^{\int 2\,dx} = e^{2x}$ so that $\frac{d}{dx}\left[e^{2x}y\right] = 0$ and $y = ce^{-2x}$ for $-\infty < x < \infty$. The transient term is ce^{-2x}.

3. For $y' + y = e^{3x}$ an integrating factor is $e^{\int dx} = e^{x}$ so that $\frac{d}{dx}\left[e^{x}y\right] = e^{4x}$ and $y = \frac{1}{4}e^{3x} + ce^{-x}$ for $-\infty < x < \infty$. The transient term is ce^{-x}.

4. For $y' + 4y = \frac{4}{3}$ an integrating factor is $e^{\int 4\,dx} = e^{4x}$ so that $\frac{d}{dx}\left[e^{4x}y\right] = \frac{4}{3}e^{4x}$ and $y = \frac{1}{3} + ce^{-4x}$ for $-\infty < x < \infty$. The transient term is ce^{-4x}.

5. For $y' + 3x^2y = x^2$ an integrating factor is $e^{\int 3x^2\,dx} = e^{x^3}$ so that $\frac{d}{dx}\left[e^{x^3}y\right] = x^2e^{x^3}$ and $y = \frac{1}{3} + ce^{-x^3}$ for $-\infty < x < \infty$. The transient term is ce^{-x^3}.

6. For $y' + 2xy = x^3$ an integrating factor is $e^{\int 2x\,dx} = e^{x^2}$ so that $\frac{d}{dx}\left[e^{x^2}y\right] = x^3e^{x^2}$ and $y = \frac{1}{2}x^2 - \frac{1}{2} + ce^{-x^2}$ for $-\infty < x < \infty$. The transient term is ce^{-x^2}.

7. For $y' + \frac{1}{x}y = \frac{1}{x^2}$ an integrating factor is $e^{\int (1/x)dx} = x$ so that $\frac{d}{dx}\left[xy\right] = \frac{1}{x}$ and $y = \frac{1}{x}\ln x + \frac{c}{x}$ for $0 < x < \infty$.

8. For $y' - 2y = x^2 + 5$ an integrating factor is $e^{-\int 2\,dx} = e^{-2x}$ so that $\frac{d}{dx}\left[e^{-2x}y\right] = x^2e^{-2x} + 5e^{-2x}$ and $y = -\frac{1}{2}x^2 - \frac{1}{2}x - \frac{11}{4} + ce^{2x}$ for $-\infty < x < \infty$.

9. For $y' - \frac{1}{x}y = x\sin x$ an integrating factor is $e^{-\int (1/x)dx} = \frac{1}{x}$ so that $\frac{d}{dx}\left[\frac{1}{x}y\right] = \sin x$ and $y = cx - x\cos x$ for $0 < x < \infty$.

10. For $y' + \frac{2}{x}y = \frac{3}{x}$ an integrating factor is $e^{\int (2/x)dx} = x^2$ so that $\frac{d}{dx}\left[x^2y\right] = 3x$ and $y = \frac{3}{2} + cx^{-2}$ for $0 < x < \infty$.

11. For $y' + \frac{4}{x}y = x^2 - 1$ an integrating factor is $e^{\int (4/x)dx} = x^4$ so that $\frac{d}{dx}\left[x^4y\right] = x^6 - x^4$ and $y = \frac{1}{7}x^3 - \frac{1}{5}x + cx^{-4}$ for $0 < x < \infty$.

Exactas

1. Let $M = 2x - 1$ and $N = 3y + 7$ so that $M_y = 0 = N_x$. From $f_x = 2x - 1$ we obtain $f = x^2 - x + h(y)$, $h'(y) = 3y + 7$, and $h(y) = \frac{3}{2}y^2 + 7y$. A solution is $x^2 - x + \frac{3}{2}y^2 + 7y = c$.

2. Let $M = 2x + y$ and $N = -x - 6y$. Then $M_y = 1$ and $N_x = -1$, so the equation is not exact.

3. Let $M = 5x + 4y$ and $N = 4x - 8y^3$ so that $M_y = 4 = N_x$. From $f_x = 5x + 4y$ we obtain $f = \frac{5}{2}x^2 + 4xy + h(y)$, $h'(y) = -8y^3$, and $h(y) = -2y^4$. A solution is $\frac{5}{2}x^2 + 4xy - 2y^4 = c$.

4. Let $M = \sin y - y\sin x$ and $N = \cos x + x\cos y - y$ so that $M_y = \cos y - \sin x = N_x$. From $f_x = \sin y - y\sin x$ we obtain $f = x\sin y + y\cos x + h(y)$, $h'(y) = -y$, and $h(y) = -\frac{1}{2}y^2$. A solution is $x\sin y + y\cos x - \frac{1}{2}y^2 = c$.

5. Let $M = 2y^2x - 3$ and $N = 2yx^2 + 4$ so that $M_y = 4xy = N_x$. From $f_x = 2y^2x - 3$ we obtain $f = x^2y^2 - 3x + h(y)$, $h'(y) = 4$, and $h(y) = 4y$. A solution is $x^2y^2 - 3x + 4y = c$.

6. Let $M = 4x^3 - 3y\sin 3x - y/x^2$ and $N = 2y - 1/x + \cos 3x$ so that $M_y = -3\sin 3x - 1/x^2$ and $N_x = 1/x^2 - 3\sin 3x$. The equation is not exact.

7. Let $M = x^2 - y^2$ and $N = x^2 - 2xy$ so that $M_y = -2y$ and $N_x = 2x - 2y$. The equation is not exact.

8. Let $M = 1 + \ln x + y/x$ and $N = -1 + \ln x$ so that $M_y = 1/x = N_x$. From $f_y = -1 + \ln x$ we obtain $f = -y + y\ln x + h(y)$, $h'(x) = 1 + \ln x$, and $h(y) = x\ln x$. A solution is $-y + y\ln x + x\ln x = c$.

Homogeneas

1. Letting $y = ux$ we have

$$\begin{aligned}(x - ux)\,dx + x(u\,dx + x\,du) &= 0\\ dx + x\,du &= 0\\ \frac{dx}{x} + du &= 0\\ \ln|x| + u &= c\\ x\ln|x| + y &= cx.\end{aligned}$$

2. Letting $y = ux$ we have

$$\begin{aligned}(x + ux)\,dx + x(u\,dx + x\,du) &= 0\\ (1 + 2u)\,dx + x\,du &= 0\\ \frac{dx}{x} + \frac{du}{1 + 2u} &= 0\\ \ln|x| + \frac{1}{2}\ln|1 + 2u| &= c\\ x^2\left(1 + 2\frac{y}{x}\right) &= c_1\\ x^2 + 2xy &= c_1.\end{aligned}$$

3. Letting $x = vy$ we have

$$\begin{aligned}vy(v\,dy + y\,dv) + (y - 2vy)\,dy &= 0\\ vy^2\,dv + y\left(v^2 - 2v + 1\right)dy &= 0\\ \frac{v\,dv}{(v-1)^2} + \frac{dy}{y} &= 0\\ \ln|v - 1| - \frac{1}{v - 1} + \ln|y| &= c\\ \ln\left|\frac{x}{y} - 1\right| - \frac{1}{x/y - 1} + \ln y &= c\\ (x - y)\ln|x - y| - y &= c(x - y).\end{aligned}$$

4. Letting $x = vy$ we have

$$\begin{aligned}y(v\,dy + y\,dv) - 2(vy + y)\,dy &= 0\\ y\,dv - (v + 2)\,dy &= 0\\ \frac{dv}{v + 2} - \frac{dy}{y} &= 0\\ \ln|v + 2| - \ln|y| &= c\\ \ln\left|\frac{x}{y} + 2\right| - \ln|y| &= c\\ x + 2y &= c_1 y^2.\end{aligned}$$

8. Letting $y = ux$ we have

$$\begin{aligned} (x + 3ux)\,dx - (3x + ux)(u\,dx + x\,du) &= 0 \\ (u^2 - 1)\,dx + x(u + 3)\,du &= 0 \\ \frac{dx}{x} + \frac{u + 3}{(u - 1)(u + 1)}\,du &= 0 \\ \ln|x| + 2\ln|u - 1| - \ln|u + 1| &= c \\ \frac{x(u - 1)^2}{u + 1} &= c_1 \\ x\left(\frac{y}{x} - 1\right)^2 &= c_1\left(\frac{y}{x} + 1\right) \\ (y - x)^2 &= c_1(y + x). \end{aligned}$$

9. Letting $y = ux$ we have

$$\begin{aligned} -ux\,dx + (x + \sqrt{u}\,x)(u\,dx + x\,du) &= 0 \\ (x^2 + x^2\sqrt{u}\,)\,du + xu^{3/2}\,dx &= 0 \\ \left(u^{-3/2} + \frac{1}{u}\right)du + \frac{dx}{x} &= 0 \\ -2u^{-1/2} + \ln|u| + \ln|x| &= c \\ \ln|y/x| + \ln|x| &= 2\sqrt{x/y} + c \\ y(\ln|y| - c)^2 &= 4x. \end{aligned}$$

Bernoulli

15. From $y' + \dfrac{1}{x}y = \dfrac{1}{x}y^{-2}$ and $w = y^3$ we obtain $\dfrac{dw}{dx} + \dfrac{3}{x}w = \dfrac{3}{x}$. An integrating factor is x^3 so that $x^3 w = x^3 + c$ or $y^3 = 1 + cx^{-3}$.

16. From $y' - y = e^x y^2$ and $w = y^{-1}$ we obtain $\dfrac{dw}{dx} + w = -e^x$. An integrating factor is e^x so that $e^x w = -\frac{1}{2}e^{2x} + c$ or $y^{-1} = -\frac{1}{2}e^x + ce^{-x}$.

17. From $y' + y = xy^4$ and $w = y^{-3}$ we obtain $\dfrac{dw}{dx} - 3w = -3x$. An integrating factor is e^{-3x} so that $e^{-3x}w = xe^{-3x} + \frac{1}{3}e^{-3x} + c$ or $y^{-3} = x + \frac{1}{3} + ce^{3x}$.

18. From $y' - \left(1 + \dfrac{1}{x}\right)y = y^2$ and $w = y^{-1}$ we obtain $\dfrac{dw}{dx} + \left(1 + \dfrac{1}{x}\right)w = -1$. An integrating factor is xe^x so that $xe^x w = -xe^x + e^x + c$ or $y^{-1} = -1 + \dfrac{1}{x} + \dfrac{c}{x}e^{-x}$.

19. From $y' - \dfrac{1}{t}y = -\dfrac{1}{t^2}y^2$ and $w = y^{-1}$ we obtain $\dfrac{dw}{dt} + \dfrac{1}{t}w = \dfrac{1}{t^2}$. An integrating factor is t so that $tw = \ln t + c$ or $y^{-1} = \dfrac{1}{t}\ln t + \dfrac{c}{t}$. Writing this in the form $\dfrac{t}{y} = \ln t + c$, we see that the solution can also be expressed in the form $e^{t/y} = c_1 t$.

20. From $y' + \dfrac{2}{3(1+t^2)}y = \dfrac{2t}{3(1+t^2)}y^4$ and $w = y^{-3}$ we obtain $\dfrac{dw}{dt} - \dfrac{2t}{1+t^2}w = \dfrac{-2t}{1+t^2}$. An integrating factor is $\dfrac{1}{1+t^2}$ so that $\dfrac{w}{1+t^2} = \dfrac{1}{1+t^2} + c$ or $y^{-3} = 1 + c\left(1+t^2\right)$.

21. From $y' - \dfrac{2}{x}y = \dfrac{3}{x^2}y^4$ and $w = y^{-3}$ we obtain $\dfrac{dw}{dx} + \dfrac{6}{x}w = -\dfrac{9}{x^2}$. An integrating factor is x^6 so that $x^6 w = -\frac{9}{5}x^5 + c$ or $y^{-3} = -\frac{9}{5}x^{-1} + cx^{-6}$. If $y(1) = \frac{1}{2}$ then $c = \frac{49}{5}$ and $y^{-3} = -\frac{9}{5}x^{-1} + \frac{49}{5}x^{-6}$.